PATTERN RECOGNITION ON ORIENTED MATROIDS: κ^*-VECTORS AND REORIENTATIONS

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Abstract. The components of κ^*-vectors associated to a simple oriented matroid $\mathcal{M}$ are the numbers of general or special tope committees for $\mathcal{M}$. Using the principle of inclusion-exclusion, we determine how the reorientations of $\mathcal{M}$ on one-element subsets of its ground set affect κ^*-vectors.

Contents

1. Introduction

Let $\mathcal{M} := (E_t, \mathcal{T})$ be a simple oriented matroid on the ground set $E_t := \{1, \ldots, t\}$, with set of topes $\mathcal{T}$; throughout we will suppose that it is *simple*, that is, it contains no loops, parallel or *antiparallel* elements.

See, e.g., [2, 3, 4, 5, 12, 13, 15] on oriented matroids.

Associated to each element $e \in E_t$ are the corresponding *positive halfspace* $\mathcal{T}_e^+ := \{T \in \mathcal{T} : T(e) = +\}$ and *negative halfspace* $\mathcal{T}_e^- := \{T \in \mathcal{T} : T(e) = -\}$ of $\mathcal{M}$. If $\mathcal{T}_e^\bullet \subset \mathcal{T}$ is a halfspace of $\mathcal{M}$ then we denote by $\binom{\mathcal{T}_e^\bullet}{j}$ the family of j-subsets of the set $\mathcal{T}_e^\bullet$.

If $G \subseteq \mathcal{T}$ is a subset of topes then $-G$ stands for the set of their opposites $\{-T : T \in G\}$.

If $A \subseteq E_t$ then ${}_{-A}\mathcal{M}$ denotes the oriented matroid obtained from $\mathcal{M}$ by *reorientation* on the set A; if $a \in E_t$ then we write ${}_{-a}\mathcal{M}$ instead of ${}_{-\{a\}}\mathcal{M}$.

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A subset $\mathcal{K}^* \subset \mathcal{T}$ is called a *tope committee* for $\mathcal{M}$ if for each element $e \in E_t$ it holds

$$|\{T \in \mathcal{K}^* : \ T(e) = +\}| > \tfrac{1}{2}|\mathcal{K}^*| \ ,$$

see [7, 8, 9, 10]; in other words, if we replace the components $-$ and $+$ of the maximal covectors of the oriented matroid $\mathcal{M}$ by the real numbers -1 and 1, respectively, then a collection $\mathcal{K}^* \subset \mathcal{T}$ is a committee for $\mathcal{M}$ iff the strict inequality

$$\sum_{T \in \mathcal{K}^*} T > \mathbf{0}$$

holds componentwise.

Let $\mathbf{K}^*_k(\mathcal{M})$ denote the family of tope committees, of cardinality k, for $\mathcal{M}$, and let $\mathbf{K}^*(\mathcal{M}) := \dot{\bigcup}_{1 \le k \le |\mathcal{T}|-1} \mathbf{K}^*_k(\mathcal{M})$ denote the family of all tope committees for $\mathcal{M}$. By definition, the kth component $\kappa^*_k(\mathcal{M}) := \#\mathbf{K}^*_k(\mathcal{M})$ of the vector $\boldsymbol{\kappa}^*(\mathcal{M}) \in \mathbb{N}^{|\mathcal{T}|/2}$, $1 \le k \le |\mathcal{T}|/2$, is the number of committees in the family $\mathbf{K}^*_k(\mathcal{M})$.

Similarly, we associate to each family $\mathring{\mathbf{K}}^*_k(\mathcal{M})$, $1 \le k \le |\mathcal{T}|/2$, of tope committees, of cardinality k, that contain no pairs of opposites, the kth component $\mathring{\kappa}^*_k(\mathcal{M}) := \#\mathring{\mathbf{K}}^*_k(\mathcal{M})$ of the vector $\mathring{\boldsymbol{\kappa}}^*(\mathcal{M}) \in \mathbb{N}^{|\mathcal{T}|/2}$.

We always have $\mathring{\kappa}^*_2(\mathcal{M}) = \kappa^*_2(\mathcal{M}) = 0$. The oriented matroid $\mathcal{M}$ is acyclic iff $\mathring{\kappa}^*_1(\mathcal{M}) = \kappa^*_1(\mathcal{M}) = 1$. If $\mathcal{M}$ is not acyclic then $\mathring{\kappa}^*_1(\mathcal{M}) = \kappa^*_1(\mathcal{M}) = 0$ and $\mathring{\kappa}^*_3(\mathcal{M}) = \kappa^*_3(\mathcal{M})$.

If $\mathcal{K}^* \in \mathring{\mathbf{K}}^*_j(\mathcal{M})$, for some j, $1 \le j \le |\mathcal{T}|/2$, then there are $|\mathcal{T}|/2 - j$ pairs of topes $\{T, -T\} \subset \mathcal{T}$ such that $|\mathcal{K}^* \cap \{T, -T\}| = 0$. If we add any such pairs of opposites to the set $\mathcal{K}^*$ then the resulting set is a committee for $\mathcal{M}$. Thus, given an integer k such that $j \le k \le |\mathcal{T}|/2$ and the difference $k - j$ is even, in the family $\mathbf{K}^*_k(\mathcal{M})$ there are exactly $\binom{(|\mathcal{T}|-2j)/2}{(k-j)/2}$ tope committees which contain the committee $\mathcal{K}^*$ as a subset. We see that

$$\kappa^*_k(\mathcal{M}) = \sum_{\substack{1 \le j \le k: \\ j \equiv k \pmod 2}} \binom{(|\mathcal{T}| - 2j)/2}{(k-j)/2} \cdot \mathring{\kappa}^*_j(\mathcal{M}) \ , \quad 1 \le k \le |\mathcal{T}|/2 \ ;$$

for example, $\kappa^*_3(\mathcal{M}) = \frac{|\mathcal{T}|-2}{2} \cdot \mathring{\kappa}^*_1(\mathcal{M}) + \mathring{\kappa}^*_3(\mathcal{M})$, and $\kappa^*_5(\mathcal{M}) = \frac{(|\mathcal{T}|-4)(|\mathcal{T}|-2)}{8} \cdot \mathring{\kappa}^*_1(\mathcal{M}) + \frac{|\mathcal{T}|-6}{2} \cdot \mathring{\kappa}^*_3(\mathcal{M}) + \mathring{\kappa}^*_5(\mathcal{M})$.

The family $\mathbf{A}^*(\mathcal{M})$ of *anti-committees* for the oriented matroid $\mathcal{M}$ is defined as the family $\{-\mathcal{K}^* : \ \mathcal{K}^* \in \mathbf{K}^*(\mathcal{M})\}$.

Let A be any subset of the ground set E_t. The tope sets of the oriented matroids ${}_{-A}\mathcal{M}$ and ${}_{-(E_t - A)}\mathcal{M}$ coincide and, thanks to the composite bijection

$$\begin{array}{ccccccc} \mathbf{K}^*({}_{-A}\mathcal{M}) & \to & \mathbf{A}^*({}_{-A}\mathcal{M}) & \to & \mathbf{A}^*({}_{-(E_t-A)}\mathcal{M}) & \to & \mathbf{K}^*({}_{-(E_t-A)}\mathcal{M}) \ , \\ \mathcal{K}^* & \mapsto & -\mathcal{K}^* & \mapsto & -\mathcal{K}^* & \mapsto & \mathcal{K}^* \ , \end{array}$$

the (anti-)committee structures of ${}_{-A}\mathcal{M}$ and ${}_{-(E_t-A)}\mathcal{M}$ are identical; in particular, we have

$$\boldsymbol{\kappa}^*({}_{-A}\mathcal{M}) = \boldsymbol{\kappa}^*({}_{-(E_t-A)}\mathcal{M})$$

and

$$\mathring{\boldsymbol{\kappa}}^*({}_{-A}\mathcal{M}) = \mathring{\boldsymbol{\kappa}}^*({}_{-(E_t-A)}\mathcal{M}) \ .$$

In this paper we compare κ^*-vectors of the oriented matroids $\mathcal{M}$ and ${}_{-A}\mathcal{M}$, where $A := \{a\}$ are one-element subsets of the ground set E_t. In Section 4 we sum up the observations that concern general tope committees and committees containing no pairs of opposites, made in Sections 2 and 3, respectively.

2. The Number of Tope Committees

Consider general tope committees for the oriented matroid $\mathcal{M}$ and begin by restating expression [7, (3.2)]:

Lemma 2.1. *The number $\#\mathbf{K}_k^*(\mathcal{M})$ of tope committees, of cardinality k, $1 \le k \le |\mathcal{T}| - 1$, for the oriented matroid $\mathcal{M} := (E_t, \mathcal{T})$, is*

$$\#\mathbf{K}_k^*(\mathcal{M}) = \binom{|\mathcal{T}|}{|\mathcal{T}|-\ell} + \sum_{\substack{\mathcal{G} \subseteq \bigcup_{e \in E_t} \binom{\mathcal{T}_e^+}{\lfloor (\ell+1)/2 \rfloor}: \\ 1 \le \#\mathcal{G} \le \binom{\ell}{\lfloor (\ell+1)/2 \rfloor}, \\ |\bigcup_{G \in \mathcal{G}} G| \le \ell}} (-1)^{\#\mathcal{G}} \cdot \binom{|\mathcal{T}| - |\bigcup_{G \in \mathcal{G}} G|}{|\mathcal{T}| - \ell} , \tag{2.1}$$

where $\ell \in \{k, |\mathcal{T}| - k\}$.

Fix an integer k, $1 \le k \le |\mathcal{T}|/2$, a ground element $a \in E_t$, and an integer $\ell \in \{k, |\mathcal{T}| - k\}$. If we set

$$\alpha_k(a, \mathcal{M}) := \binom{|\mathcal{T}|}{|\mathcal{T}|-\ell} + \sum_{\substack{\mathcal{G} \subseteq \bigcup_{e \in E_t - \{a\}} \binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor (\ell+1)/2 \rfloor}: \\ 1 \le \#\mathcal{G} \le \binom{\ell}{\lfloor (\ell+1)/2 \rfloor}, \\ |\bigcup_{G \in \mathcal{G}} G| \le \ell}} (-1)^{\#\mathcal{G}} \cdot \binom{|\mathcal{T}| - |\bigcup_{G \in \mathcal{G}} G|}{|\mathcal{T}| - \ell}$$

then, according to (2.1), we have

$$\begin{aligned} \kappa_k^*(\mathcal{M}) &= \alpha_k(a, \mathcal{M}) \\ &+ \sum_{\substack{\mathcal{G}' \subseteq \binom{\mathcal{T}_a^+(\mathcal{M})}{\lfloor (\ell+1)/2 \rfloor} - \bigcup_{e \in E_t - \{a\}} \binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor (\ell+1)/2 \rfloor}: \ 1 \le \#\mathcal{G}' \le \binom{\ell}{\lfloor (\ell+1)/2 \rfloor}, \ |\bigcup_{G \in \mathcal{G}'} G| \le \ell, \\ \mathcal{G}'' \subseteq \bigcup_{e \in E_t - \{a\}} \binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor (\ell+1)/2 \rfloor}: \ 0 \le \#\mathcal{G}'' \le \binom{\ell}{\lfloor (\ell+1)/2 \rfloor} - \#\mathcal{G}', \ |\bigcup_{G \in \mathcal{G}' \dot\cup \mathcal{G}''} G| \le \ell}} (-1)^{\#\mathcal{G}' + \#\mathcal{G}''} \\ &\cdot \binom{|\mathcal{T}| - |\bigcup_{G \in \mathcal{G}' \dot\cup \mathcal{G}''} G|}{|\mathcal{T}| - \ell} . \end{aligned} \tag{2.2}$$

In an analogous expression for $\kappa_k^*({}_{-a}\mathcal{M})$ the families $\mathcal{G}'$ range over subfamilies of the family $\binom{\mathcal{T}_a^-(\mathcal{M})}{\lfloor (\ell+1)/2 \rfloor} - \bigcup_{e \in E_t - \{a\}} \binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor (\ell+1)/2 \rfloor}$.

3. The Number of Tope Committees Containing no Pairs of Opposites

Before proceeding to consider the tope committees that contain no pairs of opposites, we collect a few observations:

Let m be a positive integer, and $\pm[1,m]$ the $2m$-set $\{-m,\ldots,-1,1,\ldots,m\}$. If we fix a subset $W\subseteq\pm[1,m]$ and denote by $-W$ the set $\{-w:\ w\in W\}$ then we have

$$|\pm[1,m]|-|W|-2\#\bigl\{\{i,-i\}\subseteq\pm[1,m]:\ |\{i,-i\}\cap W|=0\bigr\}\\ =|W\cup -W|-|W| \quad (3.1)$$

and

$$\#\bigl\{\{i,-i\}\subseteq\pm[1,m]:\ |\{i,-i\}\cap W|=0\bigr\}=m-\tfrac{1}{2}|W\cup -W|\ . \quad (3.2)$$

Recall that the number of k-subsets $V\subset\pm[1,m]$, such that

$$v\in V\quad\Longrightarrow\quad -v\notin V\ , \quad (3.3)$$

is $\binom{m}{k}2^k$ — this is the number of $(k-1)$-dimensional faces of an m-dimensional *crosspolytope*, see [6].

If $W\neq\pm[1,m]$ then consider some nonempty k-set $V\subset\pm[1,m]$ such that $|V\cap W|=0$ and implication (3.3) holds. Let $V=V'\dot{\cup}V''$ be the partition of V into two subsets with the following properties:

$$v'\in V'\quad\Longrightarrow\quad -v'\in W\ , \quad (3.4)$$

$$v''\in V''\quad\Longrightarrow\quad -v''\notin W\ . \quad (3.5)$$

Let $|V'|=:j$ and $|V''|=:k-j$, for some j. In fact, (3.1) and (3.2) imply that there are $\binom{|W\cup-W|-|W|}{j}$ sets $V'\subset\pm[1,m]$ such that $|V'|=j$, $|V'\cap W|=0$ and (3.4) holds; there are $\binom{m-\frac{1}{2}|W\cup-W|}{k-j}2^{k-j}$ sets $V''\subset\pm[1,m]$ such that $|V''|=k-j$, $|V''\cap W|=0$ and (3.5) holds.

Let $\mathbb{B}(2m)$ denote the Boolean lattice of subsets of the set $\pm[1,m]$. The empty subset of $\pm[1,m]$ is denoted by $\hat{0}$. If $b\in\mathbb{B}(2m)-\{\hat{0}\}$ then we let $-b$ denote the set of the negations of elements from b.

Let r be a rational number, $0\leq r<1$, and k an integer number, $1\leq k\leq m$. If Λ is an antichain in $\mathbb{B}(2m)$, such that $\lfloor r\cdot k\rfloor+1\leq\min_{\lambda\in\Lambda}\rho(\lambda)$, then consider the subset

$$\mathring{\mathbf{I}}_{r,k}\bigl(\mathbb{B}(2m),\Lambda\bigr):=\bigl\{b\in\mathbb{B}(2m):\\ \rho(b)=k,\ b\wedge -b=\hat{0},\ \rho(b\wedge\lambda)>r\cdot k\quad\forall\lambda\in\Lambda\bigr\}\subset\mathbb{B}(2m)^{(k)}\ ,$$

where $\rho(\cdot)$ denotes the poset rank of an element in $\mathbb{B}(2m)$, and $\mathbb{B}(2m)^{(k)}:=\{b\in\mathbb{B}(2m):\ \rho(b)=k\}$. The collection $\mathring{\mathbf{I}}_{r,k}\bigl(\mathbb{B}(2m),\Lambda\bigr)$ is the set of *relatively r-blocking elements $b\in\mathbb{B}(2m)^{(k)}$ (with the additional property $b\wedge-b=\hat{0}$)* for the antichain Λ in the lattice $\mathbb{B}(2m)$; relative blocking is discussed in [11].

Denote by $\mathfrak{I}(\lambda)$ the principal order ideal of the lattice $\mathbb{B}(2m)$ generated by an element $\lambda \in \mathit{\Lambda}$. Using the principle of inclusion-exclusion [1, 14], we obtain

$$\begin{aligned}\left|\mathring{\mathbf{I}}_{r,k}\big(\mathbb{B}(2m),\mathit{\Lambda}\big)\right| = \binom{m}{k}2^k + \sum_{D\subseteq \mathbf{min}\bigcup_{\lambda\in\mathit{\Lambda}}(\mathbb{B}(2m)^{(\rho(\lambda)-\lfloor r\cdot k\rfloor)}\cap\mathfrak{I}(\lambda)):\ |D|>0} \\ (-1)^{|D|}\cdot\sum_{0\leq j\leq k}\binom{\rho(\bigvee_{d\in D} d\vee -\bigvee_{d\in D} d)-\rho(\bigvee_{d\in D} d)}{j} \\ \cdot\binom{m-\frac{1}{2}\rho(\bigvee_{d\in D} d\vee -\bigvee_{d\in D} d)}{k-j}2^{k-j}\ , \quad (3.6)\end{aligned}$$

where $\mathbf{min}\,\cdot$ denotes the set of minimal elements of a subposet.

Consider the lattice

$$\mathcal{E} := \Big\{\bigvee_{d\in D} d:\ D\subseteq \mathbf{min}\bigcup_{\lambda\in\mathit{\Lambda}}\big(\mathbb{B}(2m)^{(\rho(\lambda)-\lfloor r\cdot k\rfloor)}\cap\mathfrak{I}(\lambda)\big),\ |D|>0\Big\}\ \dot\cup\ \{\hat{0}\}\ ,$$

where $\hat{0}$ is a new least element adjoined. If we let $\mu_{\mathcal{E}}(\cdot,\cdot)$ denote the *Möbius function* of the lattice $\mathcal{E}$, then we have

$$\begin{aligned}\left|\mathring{\mathbf{I}}_{r,k}\big(\mathbb{B}(2m),\mathit{\Lambda}\big)\right| = \binom{m}{k}2^k + \sum_{z\in\mathcal{E}:\ z>\hat{0}}\mu_{\mathcal{E}}(\hat{0},z) \\ \cdot\sum_{0\leq j\leq k}\binom{\rho(z\vee -z)-\rho(z)}{j}\binom{m-\frac{1}{2}\rho(z\vee -z)}{k-j}2^{k-j}\ , \quad (3.7)\end{aligned}$$

where $\rho(z)$ denotes the poset rank of an element z in the lattice $\mathbb{B}(2m)$.

It was shown in [7] that any tope committee $\mathcal{K}^*\in\mathbf{K}_k^*(\mathcal{M})$ for the oriented matroid $\mathcal{M}$ is a blocking k-set for the family $\bigcup_{e\in E_t}\binom{\mathcal{T}_e^+}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}$ of tope subsets, of cardinality $\lfloor(|\mathcal{T}|-k+1)/2\rfloor$, each of which is contained in some positive halfspace, see Lemma 2.1. As a consequence, the subfamily $\mathring{\mathbf{K}}_k^*(\mathcal{M})\subset\mathbf{K}_k^*(\mathcal{M})$ is precisely the collection of blocking k-sets, that are free of opposites, for the family $\bigcup_{e\in E_t}\binom{\mathcal{T}_e^+}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}$. With the help of (3.6), we come to the following conclusion:

Lemma 3.1. *The number* $\#\overset{\circ}{\mathbf{K}}{}^*_k(\mathcal{M})$ *of tope committees, of cardinality* k, $1 \leq k \leq |\mathcal{T}|/2$, *that contain no pairs of opposites, for the oriented matroid* $\mathcal{M} := (E_t, \mathcal{T})$, *is*

$$\#\overset{\circ}{\mathbf{K}}{}^*_k(\mathcal{M}) = \binom{|\mathcal{T}|/2}{k} 2^k + \sum_{\substack{\mathcal{G} \subseteq \bigcup_{e \in E_t} \binom{\mathcal{T}_e^+}{\lfloor (|\mathcal{T}|-k+1)/2 \rfloor}: \\ 1 \leq \#\mathcal{G} \leq \binom{|\mathcal{T}|-k}{\lfloor (|\mathcal{T}|-k+1)/2 \rfloor}, \\ |\bigcup_{G \in \mathcal{G}} G| \leq |\mathcal{T}|-k}} (-1)^{\#\mathcal{G}}$$
$$\cdot \sum_{0 \leq j \leq k} \binom{|\bigcup_{G\in\mathcal{G}} G \ \cup \ -\bigcup_{G\in\mathcal{G}} G| - |\bigcup_{G\in\mathcal{G}} G|}{j}$$
$$\cdot \binom{\frac{1}{2}(\ |\mathcal{T}| - |\bigcup_{G\in\mathcal{G}} G \ \cup \ -\bigcup_{G\in\mathcal{G}} G|\ )}{k-j} 2^{k-j} \ . \qquad (3.8)$$

If $\mathcal{G}$ is a family of tope subsets then we denote by $\boldsymbol{\mathcal{E}}(\mathcal{G})$ the join-semilattice $\{\bigcup_{F\in\mathcal{F}} F : \ \mathcal{F} \subseteq \mathcal{G}, \ \#\mathcal{F} > 0\}$ that consists of the unions of the sets from the family $\mathcal{G}$ ordered by inclusion and augmented by a new least element $\hat{0}$ which is interpreted as the empty set. The Möbius function of the lattice $\boldsymbol{\mathcal{E}}(\mathcal{G})$ is denoted by $\mu_{\boldsymbol{\mathcal{E}}}(\cdot,\cdot)$.

With the help of (3.7), Lemma 3.1 can be restated in the following way:

Proposition 3.2. *The number* $\#\overset{\circ}{\mathbf{K}}{}^*_k(\mathcal{M})$ *of tope committees which are free of opposites, of cardinality* k, $1 \leq k \leq |\mathcal{T}|/2$, *for the oriented matroid* $\mathcal{M} := (E_t, \mathcal{T})$, *is:*

$$\#\overset{\circ}{\mathbf{K}}{}^*_k(\mathcal{M}) = \binom{|\mathcal{T}|/2}{k} 2^k + \sum_{G \in \boldsymbol{\mathcal{E}}(\bigcup_{e\in E_t} \binom{\mathcal{T}_e^+}{\lfloor (|\mathcal{T}|-k+1)/2 \rfloor}): \ 0 < |G| \leq |\mathcal{T}|-k} \mu_{\boldsymbol{\mathcal{E}}}(\hat{0}, G)$$
$$\cdot \sum_{0 \leq j \leq k} \binom{|G \ \cup \ -G| - |G|}{j} \binom{\frac{1}{2}(\ |\mathcal{T}| - |G \ \cup \ -G|\ )}{k-j} 2^{k-j} \ .$$

If an integer k, $1 \leq k \leq |\mathcal{T}|/2$, and a ground element $a \in E_t$ are fixed, then we set

$$\beta_k(a, \mathcal{M}) := \binom{|\mathcal{T}|/2}{k} 2^k + \sum_{\substack{\mathcal{G} \subseteq \bigcup_{e \in E_t - \{a\}} \binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor (|\mathcal{T}|-k+1)/2 \rfloor}: \\ 1 \leq \#\mathcal{G} \leq \binom{|\mathcal{T}|-k}{\lfloor (|\mathcal{T}|-k+1)/2 \rfloor}, \\ |\bigcup_{G \in \mathcal{G}} G| \leq |\mathcal{T}|-k}} (-1)^{\#\mathcal{G}}$$
$$\cdot \sum_{0 \leq j \leq k} \binom{|\bigcup_{G\in\mathcal{G}} G \ \cup \ -\bigcup_{G\in\mathcal{G}} G| - |\bigcup_{G\in\mathcal{G}} G|}{j}$$
$$\cdot \binom{\frac{1}{2}(\ |\mathcal{T}| - |\bigcup_{G\in\mathcal{G}} G \ \cup \ -\bigcup_{G\in\mathcal{G}} G|\ )}{k-j} 2^{k-j} \ .$$

In view of (3.8), we have

$$\overset{\circ}{\kappa}{}^{*}_{k}(\mathcal{M}) = \beta_k(a,\mathcal{M})$$
$$+ \sum_{\substack{\mathcal{G}' \subseteq \binom{\mathcal{T}^+_a(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor} - \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}^+_e(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}:\ 1\le\#\mathcal{G}'\le\binom{|\mathcal{T}|-k}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor},\ |\bigcup_{G\in\mathcal{G}'} G|\le|\mathcal{T}|-k,\\ \mathcal{G}'' \subseteq \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}^+_e(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}:\ 0\le\#\mathcal{G}''\le\binom{|\mathcal{T}|-k}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}-\#\mathcal{G}',\ |\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''} G|\le|\mathcal{T}|-k}} (-1)^{\#\mathcal{G}'+\#\mathcal{G}''}$$
$$\cdot \sum_{0\le j\le k} \binom{|\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''} G\ \cup\ -\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''} G| - |\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''} G|}{j}$$
$$\cdot \binom{\frac{1}{2}(\ |\mathcal{T}| - |\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''} G\ \cup\ -\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''} G|\ )}{k-j} 2^{k-j}\ . \tag{3.9}$$

In an analogous expression for $\overset{\circ}{\kappa}{}^{*}_{k}({}_{-a}\mathcal{M})$ the families $\mathcal{G}'$ range over subfamilies of the family $\binom{\mathcal{T}^-_a(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor} - \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}^+_e(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}$.

4. κ^*-Vectors and Reorientations on One-Element Sets

To find the differences of the components of κ^*-vectors associated to the oriented matroid $\mathcal{M}$ and to the oriented matroid ${}_{-a}\mathcal{M}$ which is obtained from $\mathcal{M}$ by reorientation on a one-element subset $\{a\} \subset E_t$, we combine expressions (2.2) and (3.9) related to $\mathcal{M}$ with analogous expressions related to ${}_{-a}\mathcal{M}$:

Proposition 4.1. *Let a be an element of the ground set E_t of the oriented matroid $\mathcal{M} := (E_t, \mathcal{T})$. For an integer k, $1 \le k \le |\mathcal{T}|/2$, the sum*

$$\sum_{\substack{\mathcal{G}'' \subseteq \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}^+_e(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}:\\ 0\le\#\mathcal{G}''\le\binom{|\mathcal{T}|-k}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}-1,\\ |\bigcup_{G\in\mathcal{G}''} G|\le|\mathcal{T}|-k}} (-1)^{\#\mathcal{G}''}$$
$$\cdot \Bigg(\sum_{\substack{\mathcal{G}' \subseteq \binom{\mathcal{T}^-_a(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor} - \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}^+_e(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}:\\ 1\le\#\mathcal{G}'\le\binom{|\mathcal{T}|-k}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}-\#\mathcal{G}'',\\ |\bigcup_{G\in\mathcal{G}'} G|\le|\mathcal{T}|-k,}} (-1)^{\#\mathcal{G}'} \cdot Q(\mathcal{G}',\mathcal{G}'')$$
$$- \sum_{\substack{\mathcal{G}' \subseteq \binom{\mathcal{T}^+_a(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor} - \bigcup_{e\in E_t-\{a\}} \binom{\mathcal{T}^+_e(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}:\\ 1\le\#\mathcal{G}'\le\binom{|\mathcal{T}|-k}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}-\#\mathcal{G}'',\\ |\bigcup_{G\in\mathcal{G}'} G|\le|\mathcal{T}|-k,}} (-1)^{\#\mathcal{G}'} \cdot Q(\mathcal{G}',\mathcal{G}'') \Bigg)$$

and the sum

$$\sum_{\substack{G''\in\boldsymbol{\mathcal{E}}(\bigcup_{e\in E_t-\{a\}}\binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}):\\ 0\leq|G''|\leq|\mathcal{T}|-k}}\mu_{\boldsymbol{\mathcal{E}}}(\hat{0},G'')$$

$$\cdot\Biggl(\sum_{\substack{G'\in\boldsymbol{\mathcal{E}}(\binom{\mathcal{T}_a^-(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor})-\bigcup_{e\in E_t-\{a\}}\binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}):\\ 0<|G'|\leq|\mathcal{T}|-k}}\mu_{\boldsymbol{\mathcal{E}}}(\hat{0},G')\cdot\mathfrak{Q}(G',G'')$$

$$-\sum_{\substack{G'\in\boldsymbol{\mathcal{E}}(\binom{\mathcal{T}_a^+(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor})-\bigcup_{e\in E_t-\{a\}}\binom{\mathcal{T}_e^+(\mathcal{M})}{\lfloor(|\mathcal{T}|-k+1)/2\rfloor}):\\ 0<|G'|\leq|\mathcal{T}|-k}}\mu_{\boldsymbol{\mathcal{E}}}(\hat{0},G')\cdot\mathfrak{Q}(G',G'')\Biggr)$$

both calculate the difference

$$\kappa_k^*({}_{-a}\mathcal{M})-\kappa_k^*(\mathcal{M})$$

under

$$Q(\mathcal{G}',\mathcal{G}''):=\binom{|\mathcal{T}|-|\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''}G|}{k}\quad\textit{and}\quad\mathfrak{Q}(G',G''):=\binom{|\mathcal{T}|-|G'\cup G''|}{k}\ .$$

These sums calculate the difference

$$\mathring{\kappa}_k^*({}_{-a}\mathcal{M})-\mathring{\kappa}_k^*(\mathcal{M})$$

under

$$Q(\mathcal{G}',\mathcal{G}''):=\sum_{0\leq j\leq k}\binom{|\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''}G\ \cup\ -\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''}G|-|\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''}G|}{j}$$
$$\cdot\binom{\frac{1}{2}(\ |\mathcal{T}|-|\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''}G\ \cup\ -\bigcup_{G\in\mathcal{G}'\dot\cup\mathcal{G}''}G|\ )}{k-j}2^{k-j}$$

and

$$\mathfrak{Q}(G',G''):=\sum_{0\leq j\leq k}\binom{|(G'\cup G'')\cup-(G'\cup G'')|-|G'\cup G''|}{j}$$
$$\cdot\binom{\frac{1}{2}(\ |\mathcal{T}|-|(G'\cup G'')\cup-(G'\cup G'')|\ )}{k-j}2^{k-j}\ .$$

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